3 Linear independence

Linear Independence A set of vectors $\mathcal{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ is said to be a *linearly independent* set whenever the only solution for the scalars α_i in the homogeneous equation

$$\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + ... + \alpha_n\boldsymbol{v}_n = \mathbf{0}$$

is the trivial solution $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. Whenever there is a nontrivial solution for the α's (i.e., at least one $\alpha_i \neq \mathbf{0}$), the set $\mathcal{S}$ is said to be a linearly dependent set. In other words, linearly independent sets are those that contain no dependency relationships, and linearly dependent sets are those in which at least one vector is a combination of the others. We will agree that the empty set is always linearly independent.

1. Determine which of the following sets are linearly independent. For those sets that are linearly dependent, write one of the vectors as a linear combination of the others.

(a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} \right\}$;

(b) $\{(1\ \ 2\ \ 3),\ (0\ \ 4\ \ 5),\ (0\ \ 0\ \ 6),\ (1\ \ 1\ \ 1)\}$;

2. Determine whether or not the following set of matrices is a linearly independent set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

3. Assume that $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. (a) Show that columns of A form a linearly independent set if and only if $\ker(A) = n$. (b) Show that columns of A form a linearly independent set if and only if $\mathrm{rank}(A) = n$.

4. Consider the matrix $\begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 2 \\ 6 & 3 & 2 & 2 \end{pmatrix}$. (a) Determine a maximal linearly independent subset of columns from A. (b) Determine the total number of linearly independent subsets that can be constructed using the columns of A.

5. Let $\mathcal{S} = \{0\}$ be the set containing only the zero vector. (a) Explain why $\mathcal{S}$ must be linearly dependent. (b) Explain why any set containing a zero vector must be linearly dependent.

Linear Independence and Matrices
Let A be an $m \times n$ matrix.

- Each of the following statements is equivalent to saying that the columns of A form a linearly independent set.

  ▷ $\ker(A) = \{\mathbf{0}\}$.

  ▷ $\mathrm{rank}(A) = n$.

- Each of the following statements is equivalent to saying that the rows of A form a linearly independent set.

  ▷ $\ker(A^\top) = \{\mathbf{0}\}$.

  ▷ $\mathrm{rank}(A) = m$.

- When A is a square matrix, each of the following statements is equivalent to saying that A is nonsingular.

  ▷ The columns of A form a linearly independent set.

  ▷ The rows of A form a linearly independent set.

6. Diagonal Dominance. A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is said to be diagonally dominant whenever

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad \text{for each } i = 1, 2, ..., n.$$

That is, the magnitude of each diagonal entry exceeds the sum of the magnitudes of the off-diagonal entries in the corresponding row. In 1900, Minkowski[1] discovered that all diagonally dominant matrices are nonsingular. Establish the validity of Minkowski's result.

7. Vandermonde Matrices. Matrices $V \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ of the form

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix}$$

[1]Hermann Minkowski (1864–1909) was born in Russia, but spent most of his life in Germany as a mathematician and professor at Königsberg and Göttingen. In addition to the inequality that now bears his name, he is known for providing a mathematical basis for the special theory of relativity. He died suddenly from a ruptured appendix at the age of 44.

[2]This is named in honor of the French mathematician Alexandre-Theophile Vandermonde (1735 – 1796). He made a variety of contributions to mathematics, but he is best known perhaps for being the first European to give a logically complete exposition of the theory of determinants. He is regarded by many as being the founder of that theory. However, the matrix V (and an associated determinant) named after him, by Lebesgue, does not appear in Vandermonde's published work. Vandermonde's first love was music, and he took up mathematics only after he was 35 years old. He advocated the theory that all art and music

in which $x_i \neq x_j$ for all $i \neq j$ are called Vandermonde[2] matrices. Explain why the columns in V constitute a linearly independent set whenever $n \leq m$.

8. Given a set of m points $\mathcal{S} = \{(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)\}$ in which the x_i's are distinct, explain why there is a unique polynomial

$$\ell(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + ... + \alpha_{m-1} t^{m-1}$$

of degree $m-1$ that passes through each point in $\mathcal{S}$.

9. If T is a triangular matrix in which each $t_{ii} \neq 0$, explain why the rows and columns of T must each be linearly independent sets.

Maximal Indt. Subsets If $\operatorname{rank}(A_{m\times n}) = r$, then the following statements hold.
- Any maximal independent subset of columns from A contains exactly r columns.
- Any maximal independent subset of rows from A contains exactly r rows.
- In particular, the r basic columns in A constitute one maximal independent subset of columns from A.

Basic Facts of Independence For a nonempty set of vectors $\mathcal{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ in a space $\mathcal{V}$, the following statements are true.

- If $\mathcal{S}$ contains a linearly dependent subset, then $\mathcal{S}$ itself must be linearly dependent.
- If $\mathcal{S}$ is linearly independent, then every subset of $\mathcal{S}$ is also linearly independent.
- If $\mathcal{S}$ is linearly independent and if $\boldsymbol{v} \in \mathcal{V}$, then the **extension set** $\mathcal{S}_{\text{ext}} = \mathcal{S} \cup \{\boldsymbol{v}\}$ is linearly independent if and only if $\boldsymbol{v} \notin \operatorname{span}(\mathcal{S})$.
- If $\mathcal{S} \subseteq \mathbb{R}^m$ and if $n > m$, then $\mathcal{S}$ must be linearly dependent.

10. Let $\mathcal{V}$ be the vector space of real-valued functions of a real variable, and let $\mathcal{S} = \{f_1(x), f_2(x), ..., f_n(x)\}$ be a set of functions that are $n-1$ times differentiable. The **Wronski**[3] **matrix** is defined to be

$$\begin{pmatrix} f_1(x) & f_2(x) & ... & f_n(x) \\ f_1'(x) & f_2'(x) & ... & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & ... & f_n^{(n-1)}(x) \end{pmatrix}$$

If there is at least one point $x = x_0$ such that $W(x_0)$ is nonsingular, prove that $\mathcal{S}$ must be a linearly independent set.

11. Which of the following sets of functions are linearly independent? (a) $\{1, x, x^2, x^3\}$. (b) $\{\sin x, \cos x, x \sin x\}$. (c) $\{e^x, xe^x, x^2 e^x\}$. (d) $\{\sin^2 x, \cos^2 x, \cos 2x\}$.

12. Let $\mathcal{L} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + x_2 = 0, -x_1 + 2x_2 + x_3 = 0\right\}$. Find a linearly independent spanning set for a vector space $\mathcal{L}$.

13. Let $\mathcal{V} = \mathbb{R}^n$ and let $(a_1, a_2, ..., a_n)^\top$ be some fixed vector from $\mathcal{V}$ and let

$$\mathcal{M} = \{(x_1, x_2, ..., x_n)^\top \in \mathcal{V} \,|\, a_1 x_1 + ... + a_n x_n = 0\}$$

be subspace of $\mathcal{V}$. Find a maximal linearly independent subset of $\mathcal{M}$.

14. Let $\mathbb{R}^+$ denote given vector space (set of all positive real numbers) over field $\mathbb{R}$, in which operations vector addition and scalar multiplication are defined on the following way

$$\text{vector addition: } \forall u, v \in \mathbb{R}^+ \quad u + v = uv;$$

$$\text{scalar multiplication: } \forall u \in \mathbb{R}^+, \quad \forall \alpha \in \mathbb{R} \quad \alpha u = u^\alpha.$$

Find minimal spanning set for $\mathcal{V}$. Explain your answer.

15. **(IMC 2016.)** Let k and n be positive integers. A sequence $(A_1, ..., A_k)$ of $n \times n$ real matrices is preferred by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_i A_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.

InC: 2, 4, 6, 7, 8, 9, 10. HW: 12, 13, 14, 15 + two more problems from the web page `http://osebje.famnit.upr.si/~penjic/linearnaAlgebra/`.

rested upon a general principle that could be expressed mathematically, and he claimed that almost anyone could become a composer with the aid of mathematics.

[3] This matrix is named in honor of the Polish mathematician Jozef Maria Höené Wronski (1778–1853), who studied four special forms of determinants, one of which was the determinant of the matrix that bears his name. Wronski was born to a poor family near Poznan, Poland, but he studied in Germany and spent most of his life in France. He is reported to have been an egotistical person who wrote in an exhaustively wearisome style. Consequently, almost no one read his work. Had it not been for his lone follower, Ferdinand Schweins (1780–1856) of Heidelberg, Wronski would probably be unknown today. Schweins preserved and extended Wronski's results in his own writings, which in turn received attention from others. Wronski also wrote on philosophy. While trying to reconcile Kant's metaphysics with Leibniz's calculus, Wronski developed a social philosophy called "Messianism" that was based on the belief that absolute truth could be achieved through mathematics.